26/9 MSA 3.220

- Mini & Alan 10%
- Midterm 35% week 29/10 - 2/11
- Final 55%

Definition of sequences

Problem: define the limit of sequences

What does it mean $\lim_{n \to +\infty} a_n = \ell$?

1st try: "When n is large, then a_n is close to ℓ"

2nd try: "When n is large, then the distance $d(\ell, a_n)$ is small"

3rd try: $\forall \varepsilon > 0$, if n is large then
$d(\ell, a_n) < \varepsilon$ $|\ell - a_n| < \varepsilon$

4th try: $\forall \varepsilon > 0$ $\exists n_0 (= n_0(\varepsilon))$ such that
if $n \geq n_0$ then $d(\ell, a_n) < \varepsilon$
$|\ell - a_n| < \varepsilon$

Examples $a_n = \begin{cases} 1 & n \text{ odd} \\ -1 & n \text{ even} \end{cases}$ $b_n = \frac{1}{n}$ $c_n = n^2$

Def of $\lim_{n \to +\infty} a_n = +\infty / -\infty$

Good The limit, if it exists, is unique.

Bad The limit might not exist.

Rmk: the limit only depends on "large" values of n.

Monotone sequences

Bounded sequences

Monotone convergence theorem

Proof with properties of real numbers:

every subset $X \subset \mathbb{R}$ is either unbounded or has a supremum

$$\sup X = \min \{ \alpha \in \mathbb{R} \mid a \geq x \ \forall x \in X \}$$

$s = \sup X$ has the properties

1) $\forall x \in X, \ x \leq s$

2) $\forall \varepsilon > 0 \ \exists x \in X$ such that $x > s - \varepsilon$.

Sign permanence theorem (?)

If $\lim a_n = a > 0$, then $\exists n_0 : \forall n \geq n_0 \ a_n > 0$

Analogously, if $a_n \leq 0$ for all $n \geq n_0$, then $\lim a_n \leq 0$ (if it exists).

Limit at $\pm\infty$ for real functions (?)

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Supremum + proof monotone convergence theorem

Def exponential

$$\exp(x) = \lim_{n\to+\infty} a_n$$

$a_0 = 1$

$a_1 = 1+x$

$\vdots$

$a_n = \sum_{m=0}^{n} \frac{x^m}{m!}$

Fact 1 a_n is bounded

$\Rightarrow$ a_n has limit (for $x \geq 0$ at least)

Fact 2 $\exp(x+y) = \exp(x)\exp(y)$

$\exp(ax) = \exp(x)^a$

$\Rightarrow$ $\exp(-x) = \frac{1}{\exp(x)}$ $\left(\exp(0) = 1 \right)$

Back to limits

Def of limit at $+\infty$ for a function.

- $\lim_{x\to+\infty} e^x = +\infty$
- $\lim_{x\to+\infty} x^n = \begin{cases} +\infty & n \geq 1 \\ 1 & n = 0 \\ 0 & n \leq -1 \end{cases}$
- $\lim_{x\to-\infty} e^x = 0$

Example of change of variables!

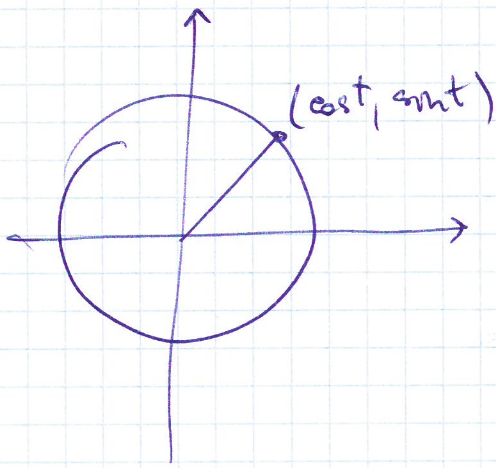
- $\lim_{x\to+\infty} \frac{e^x}{x^n} = +\infty$ for every n.
- $\lim_{x\to+\infty} x^n e^{-x} = 0$ for every n,

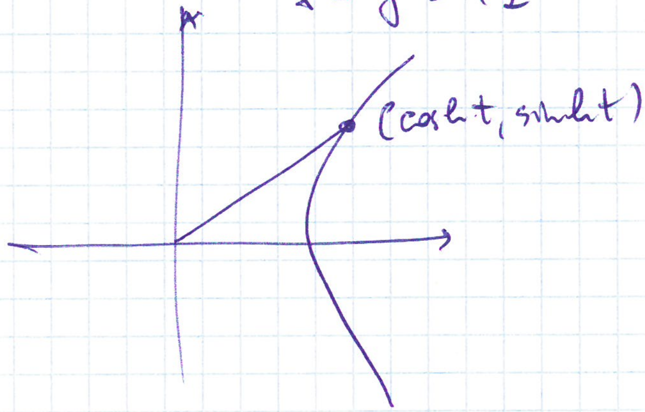
Hyperbolic trigonometric functions

As sin, cos serve to parametrize the circle $x^2+y^2=1$,

we define sinh cosh to parametrize hyperbolas

$x^2-y^2=+1$





$$\sinh t = \frac{e^t - e^{-t}}{2} \qquad \cosh t = \frac{e^t + e^{-t}}{2}$$

in fact,
$$\cosh^2 t - \sinh^2 t = \left(\frac{e^t + e^{-t}}{2}\right)^2 - \left(\frac{e^t - e^{-t}}{2}\right)^2$$
$$= \frac{2}{4} - \left(-\frac{2}{4}\right) = 1$$

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- Definition of $\lim_{x \to x_0} f(x) = \ell$.

  Examples of jump functions 1

  Definition of $\lim_{x \to x_0^\pm} f(x) = \ell$

  Examples of jump functions 2

- <u>Theorems</u>: $\lim (f+g)(x) = \lim f(x) + \lim g(x)$

  $\lim (fg)(x) = \lim f(x) \cdot \lim g(x)$

  Consequences: $\to \lim_{x \to +\infty} \frac{e^x}{p(x)} = +\infty$

  for every polynomial p.

  $\to \lim_{x \to +\infty} \frac{p(x)}{q(x)}$ according to a_m, b_m

- <u>Sandwich theorem</u>

  Théorème des gendarmes

  If $f_1(x) \leq f(x) \leq f_2(x)$ and $\lim_{x \to x_0} f_1(x) = \lim_{x \to x_0} f_2(x) = \ell$,

  then $\lim_{x \to x_0} f(x) = \ell$

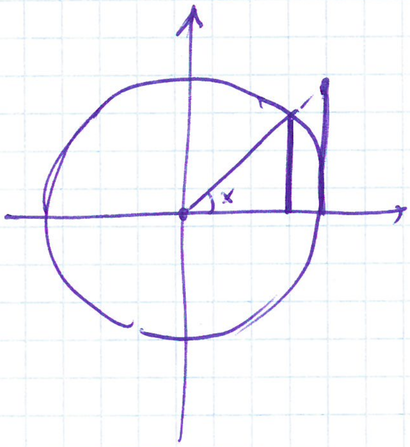
- (Maybe) <u>continuity</u>

  $\left.\begin{array}{l}\text{polynomials} \\ \text{exponential} \\ \text{trigonometric functions}\end{array}\right\}$ are continuous

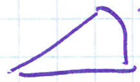
- Applications:

$\rightarrow \lim\limits_{x\to+\infty} \frac{\sin x}{x} = 0$

$\rightarrow \lim\limits_{x\to 0} \frac{\sin x}{x} = 1$



$\sin x < x < \tan x = \frac{\sin x}{\cos x}$

since area () $= \pi \cdot \frac{x}{2\pi} = \frac{x}{2}$

area () $= \frac{\tan x}{2}$

hence $\quad 1 < \frac{x}{\sin x} < \frac{1}{\cos x}$

$$\cos x \leq \frac{\sin x}{x} \leq 1$$

$$\Rightarrow \lim \frac{\sin x}{x} = 1$$

$\rightarrow \lim\limits_{x\to 0} \frac{1-\cos x}{x^2} = \lim\limits_{x\to 0} \frac{(1-\cos x)(1+\cos x)}{x^2(1+\cos x)}$

$$= \lim_{x\to 0} \frac{\sin^2 x}{x^2} \cdot \frac{1}{1+\cos x} = \frac{1}{2}$$

$\rightarrow \lim\limits_{x\to 0^+} \sin\left(\frac{1}{x}\right)$ does not exist

$\Rightarrow$ cannot extend $\sin\left(\frac{1}{x}\right)$ continuously in 0

$\rightarrow \lim\limits_{x\to 0} x \sin\left(\frac{1}{x}\right) = 0$

- Ev: <u>sign permanence theorem</u>

CONTINUITY

Definition f continuous at x_0

iff $\lim_{x \to x_0} f(x) = x$

Examples $f(x) = x$ is continuous

$f(x) \equiv a$ is continuous

Prop Sums and products of continuous functions are continuous.

- → Polynomials
- → Exponential
- → Sines and cosines
- → Tangent is continuous where defined.

Intermediate value theorem

Corollary: If f is continuous, the image of an interval is an interval.

Weierstrass theorem

if $f: [a,b] \to \mathbb{R}$ is continuous, then it has a maximum and a minimum.

Counterexample for $(a,b]$.

Ex. definition of derivative and differentiable → continuous.

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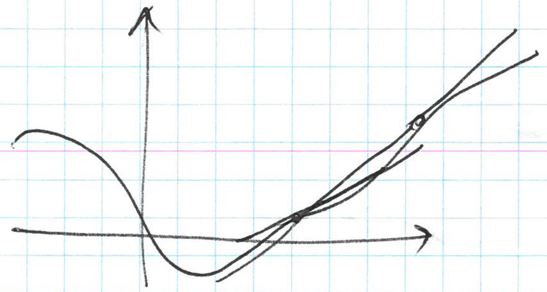
Derivatives

Secant lines: given two points x_0, x_1, the line through x_0 and x_1 is

$$f(x) = f(x_0) + a(x - x_0)$$

where $a = \dfrac{f(x_1) - f(x_0)}{x_1 - x_0}$



Def Given $f: I \to \mathbb{R}$, $x_0 \in I$, we say that f is differentiable at x_0 if the following limit exists:

$$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

In this case, we define the tangent line $L: \mathbb{R} \to \mathbb{R}$

$$L(x) = f(x_0) + f'(x_0)(x - x_0)$$

L has the property:

- $L(x_0) = f(x_0)$
- $\displaystyle \lim_{x \to x_0} \frac{f(x) - L(x)}{x - x_0} = 0$

indeed, $\dfrac{f(x) - L(x)}{x - x_0} = \dfrac{f(x) - f(x_0)}{x - x_0} - \dfrac{L(x) - f(x_0)}{x - x_0}$

$$= \frac{f(x) - f(x_0)}{x - x_0} - f'(x_0) \longrightarrow 0$$

Examples

1) $f(x) = a$
2) $f(x) = x$
3) $f(x) = ax + b$
4) $f(x) = \sin x$
$\rightarrow$ at $x_0 = 0$

$f: I \to \mathbb{R}$
if f is differentiable at x_0,
then f is continuous at x_0.

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{h \to 0} \frac{f(x_0+h) - f(x_0)}{h}$$

$$\Rightarrow \frac{\sin(x_0+h) - \sin(x_0)}{h} = \frac{\sin x_0 \cos h + \cos x_0 \sin h - \sin x_0}{h}$$

$$= \sin x_0 \underbrace{\frac{\cos h - 1}{h}}_{\to 0} + \cos x_0 \underbrace{\frac{\sin h}{h}}_{\to 1} \longrightarrow \cos x_0$$

5) $f(x) = \cos x$

$$\frac{\cos(x_0+h) - \cos x_0}{h} = \frac{\cos x_0 \cos h - \sin x_0 \sin h - \cos x_0}{h}$$

$$= \cos x_0 \frac{\cos h - 1}{h} - \sin x_0 \frac{\sin h}{h} \longrightarrow -\sin x_0$$

6) $f(x) = e^x$

$$\frac{e^{x_0+h} - e^{x_0}}{h} = \frac{e^{x_0} e^h - e^{x_0}}{h} = e^{x_0} \frac{e^h - 1}{h} \longrightarrow e^{x_0}$$

$$e^h = 1 + h + \frac{h^2}{2} + \ldots \qquad e^{h_0} - 1 = h + \frac{h^2}{2} + \frac{h^3}{6} + \ldots$$

$$\frac{e^h - 1}{h} = 1 + \frac{h}{2} + \frac{h^2}{6} + \ldots \longrightarrow 1$$

7) $f(x) = x^n$ $\quad (x_0+h)^n = x_0^n + n x_0^{n-1} h + \binom{n}{2} x_0^{n-2} h^2 + \ldots + h^n$

$$\frac{(x_0+h)^n - x_0^n}{h} = n x_0^{n-1} + \binom{n}{2} x_0^{n-2} h + \ldots + h^{n-1} \longrightarrow n x_0^{n-1}$$

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Operations with derivatives

Suppose f, g are differentiable at x_0.

- $(f+g)'(x_0) = f'(x_0) + g'(x_0)$

Indeed, $\frac{(f+g)(x)-(f+g)(x_0)}{x-x_0} = \frac{f(x)-f(x_0)}{x-x_0} + \frac{g(x)-g(x_0)}{x-x_0}$

$$\longrightarrow (f+g)'(x_0) = f'(x_0) + g'(x_0)$$

- $\left(\frac{1}{f}\right)'(x_0) = -\frac{f'(x_0)}{f(x_0)^2}$

- $(af)'(x_0) = a f'(x_0)$

Indeed, —

Indeed, $\frac{\frac{1}{f(x)} - \frac{1}{f(x_0)}}{x-x_0} = \frac{f(x_0)-f(x)}{x-x_0} \cdot \frac{1}{f(x)f(x_0)} \longrightarrow -\frac{f'(x_0)}{f(x_0)^2}$

- $(f^2)'(x_0) = 2f(x_0)f'(x_0)$

$$\frac{f(x)^2 - f(x_0)^2}{x-x_0} = \frac{(f(x)+f(x_0))(f(x)-f(x_0))}{x-x_0} \longrightarrow 2f(x_0)f'(x_0)$$

Ex: $\cos^2(x) + \sin^2(x)$
$\cos(2x) = \cos^2 x - \sin^2 x$

- $(fg)'(x_0) = f(x_0)g'(x_0) + f'(x_0)g(x_0)$

Calculate $((f+g)^2)'$

$$((f+g)^2)'(x_0) = 2(f(x_0)+g(x_0))(f'(x_0)+g'(x_0))$$

$$''$$

$$(f^2 + 2fg + g^2)' = 2f(x_0)f'(x_0) + 2(fg)'(x_0) + 2g(x_0)g'(x_0)$$

$$\Rightarrow \text{Leibnitz rule}$$

Ex: $\sin 2x = 2\sin x \cos x$

- $\left(\frac{f}{g}\right)'(x_0) = \frac{f'g(x_0) - f(x_0)g'(x_0)}{g^2(x_0)}$ if $g(x_0) \neq 0$

$\frac{f(x)}{g(x)} = f(x) \cdot \frac{1}{g(x)} \Rightarrow \left(\frac{f}{g}\right)' = f'\left(\frac{1}{g}\right) + f\left(\frac{1}{g}\right)' =$

$= \frac{f'}{g} - f\frac{g'}{g^2} = \frac{f'g - fg'}{g^2}$

Ex: tan x

- Composition: if f is differentiable at x_0
and g is differentiable at $f(x_0)$,

$$(g \circ f)'(x_0) = g'(f(x_0)) \cdot f'(x_0)$$

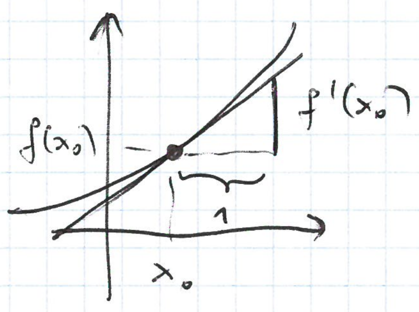
$$\frac{g(f(x)) - g(f(x_0))}{x - x_0} = \frac{g(f(x)) - g(f(x_0))}{f(x) - f(x_0)} \cdot \frac{f(x) - f(x_0)}{x - x_0}$$

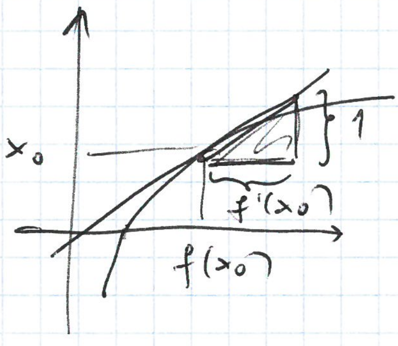
since f is continuous, $\lim_{x \to x_0} f(x) = f(x_0)$

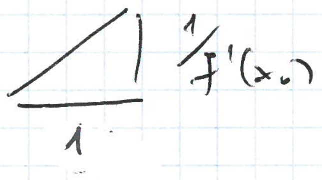
and $\lim_{x \to x_0} \frac{g(f(x)) - g(f(x_0))}{f(x) - f(x_0)} = \lim_{y \to f(x_0)} \frac{g(y) - g(f(x_0))}{y - x_0}$
$= g'(f(x_0))$

Ex cos 2x, sin 2x again

- Inverse function







hence $(f^{-1})'(y_0) = \frac{1}{f'(x_0)}$

where $y_0 = f(x_0)$

Example Logarithm, arcsin, arctan
+ power function

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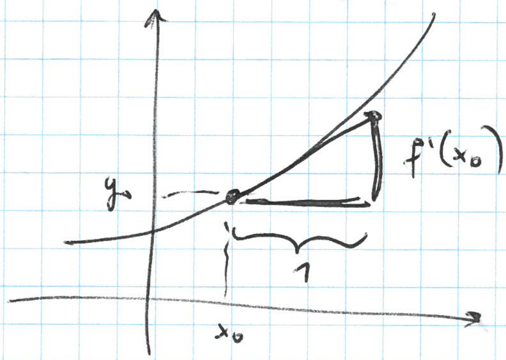
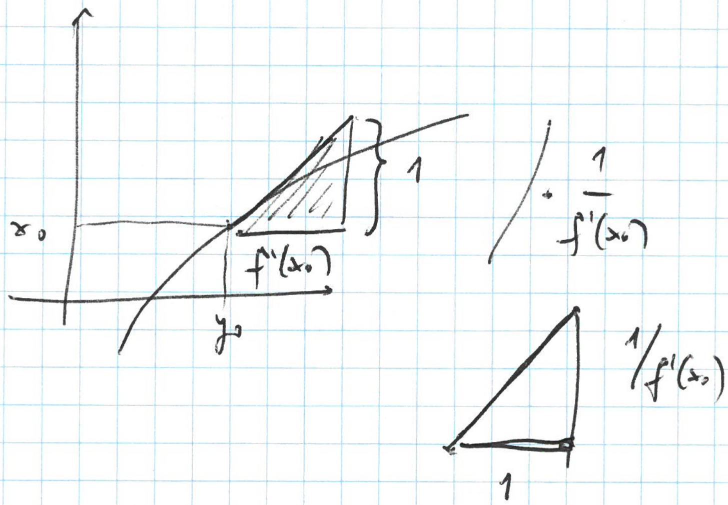
Inverse function

example: $\sqrt{x}$

graph of inverse function

$$\Rightarrow (f^{-1})'(y_0) = \frac{1}{f'(x_0)} \quad \text{where} \quad y_0 = f(x_0)$$





Indeed,

$$f^{-1} \circ f(x) = x$$

$$\Rightarrow f^{-1\,\prime}(f(x_0)) \cdot f'(x_0) = 1$$

Examples

- square root
- logarithm
- arcsin
- arccos
- arctan

More on logarithm

Properties: $\log(ab) = \log(a) + \log(b)$

Definition of power function

$$x^\alpha := e^{\alpha \log x}$$

derivative: $\frac{d}{dx}(x^\alpha) = \frac{d}{dx} e^{\alpha \log x} = \frac{\alpha}{x} e^{\alpha \log x} = \alpha e^{\alpha \log x - \log x}$

$$= \alpha e^{(\alpha-1)\log x} = \alpha x^{\alpha - 1}$$

Properties: $(e^x)^\alpha = e^{\alpha x}$

$\log y^\alpha = \alpha \log y$

$$\lim_{x \to +\infty} \frac{\log x}{x^\alpha} = 0 \qquad \forall \alpha > 0 \qquad \frac{\log x}{x^\alpha} = \frac{\log x}{e^{\alpha \log x}}$$

The logarithm goes to infinity slower than any polynomial.

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Monotonicity and minima

Definitions of (local) minimum and maximum
(strictly) monotone increasing and decreasing

- If $f: I \to \mathbb{R}$ is monotone increasing and differentiable, then $f' \geq 0$ on the interior of I

  → not true strictly increasing $\Rightarrow f' > 0$

  → moreover, if x_0 is an endpoint of I,

$$\lim_{x \to x_0^+} \frac{f(x)-f(x_0)}{x-x_0} \geq 0$$

<u>right derivative</u>

- same for f monotone decreasing

- as a consequence

<u>Theorem of Fermat</u> If x_0 is a local minimum or maximum, then $f'(x_0) = 0$.

→ The converse is not true

examples $f(x) = x^2, x^3$

Want to show that also the converse holds:

- if $f' \geq 0$ on I, then f is monotone increasing
  $f' \leq 0$ decreasing

This follows from the Theorem of Lagrange

Theorem (Lagrange)
If $f: [a,b]$ is differentiable, then $\exists \xi \in (a,b)$ such that $f'(\xi) = \dfrac{f(b) - f(a)}{b-a}$.

Remarks:

- In fact it follows that $f' \geq 0$ on $I \Rightarrow f$ monotone increasing
- moreover, it follows that
  $f' > 0$ on $I \Rightarrow f$ strictly increasing
- finally, it follows that
  $f' = 0$ on $I \Rightarrow f$ constant
  $f' = d$ on $I \Rightarrow f$ is a line

How to detect local minima and maxima

- for $f(a) = f(b)$, the theorem of Lagrange becomes

Theorem (Rolle)
if $f: [a,b]$ differentiable and $f(a) = f(b)$, then $\exists \xi \in (a,b)$ such that $f'(\xi) = 0$.

- proof of Rolle
- proof of Lagrange follows from Rolle:
  define $g(x) = f(a) + \dfrac{f(b) - f(a)}{b-a} x$ $\qquad g'(x) = \dfrac{f(b) - f(a)}{b-a}$
  then $h(x) := f(x) - g(x)$ has $h(a) = h(b) = 0$
  $\Rightarrow \exists \xi : h'(x) = 0 \Rightarrow f'(x) = g'(x) = \dfrac{f(b) - f(a)}{b-a}$.

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Convexity

Recap local minima and maxima

Sufficient condition for minimum / maximum: <u>second derivative</u>

If $f'(x_0) = 0$ and $f''(x_0) > 0 \Rightarrow$ minimum
$f''(x_0) < 0 \Rightarrow$ maximum

In fact, $f''(x_0) > 0 \Rightarrow f'(x_0) > 0$ in $(x_0, x_0+\varepsilon)$
$f'(x_0) < 0$ in $(x_0-\varepsilon, x_0)$

$\Rightarrow f$ increasing in $[x_0, x_0+\varepsilon)$
f decreasing in $(x_0-\varepsilon, x_0]$
$\Rightarrow x_0$ local minimum

<u>But</u> $f(x)=x^3$ has stationary point at 0
$f(x)=x^4$ has minimum but $f''(0) = 0$.

Moreover, second derivative tells about convexity.

$f: I \to \mathbb{R}$ is convex if

$\forall\, a, b \in I \quad \forall x \in (a,b)$

$$f(x) \le f(a) + \frac{f(b)-f(a)}{b-a}(x-a)$$
$$= f(a)\left(1 - \frac{x-a}{b-a}\right) + f(b)\frac{x-a}{b-a}$$
$$= (1-\lambda) f(a) + \lambda f(b) \qquad \lambda \in (0,1)$$

This is equivalent to

$$\frac{f(x)-f(a)}{x-a} \le \frac{f(b)-f(x)}{b-x} \qquad \circledast$$

In fact, if $\circledast$ holds, then

$$f(x)\left(\frac{1}{x-a}+\frac{1}{b-x}\right) \le \frac{f(a)}{x-a}+\frac{f(b)}{b-x}$$

$$\Rightarrow f(x)(b-x+x-a) \le f(a)(b-x)+f(b)(x-a)$$

$$\Rightarrow f(x) \le f(a)\underbrace{\frac{b-x}{b-a}}_{1-\frac{x-a}{b-a}} + f(b)\frac{x-a}{b-a}$$

<u>Prop</u> If f is differentiable on I, then f is convex

$\Longleftrightarrow$ f is increasing on I.

dim. If f is convex, then

$$\frac{f(x)-f(a)}{x-a} \le \frac{f(b)-f(a)}{b-a} \le \frac{f(b)-f(x)}{b-x}$$

$$\Rightarrow \underbrace{\lim_{x\to a}\frac{f(x)-f(a)}{x-a}}_{f'(a)} \le \frac{f(b)-f(a)}{b-a} \le \underbrace{\lim_{x\to b}\frac{f(x)-f(b)}{x-b}}_{f'(b)}$$

Conversely,

$$\frac{f(x)-f(a)}{x-a} = f'(p_1), \qquad p_1 \in (a,x)$$

$$\frac{f(b)-f(x)}{b-x} = f'(p_2), \qquad p_2 \in (x,b)$$

<u>Lagrange Theorem</u>

$$\Rightarrow f'(p_1) \le f'(p_2)$$

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Taylor series

Idea: tangent line has the properties

$$\ell(x_0) = f(x_0)$$
$$\ell'(x_0) = f'(x_0)$$

Moreover, $\dfrac{f(x) - \ell(x)}{x - x_0} \xrightarrow[x \to x_0]{} 0$

Try to use higher order polynomials to approximate a function

$$p_n(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \dots + a_n(x - x_0)^n$$

We want $p_n^{(k)}(x_0) = f^{(k)}(x_0) \quad \forall k$

$$p_n(x_0) = a_0 = f(x_0)$$
$$p_n'(x_0) = a_1 = f'(x_0)$$
$$p_n''(x_0) = 2a_2 = f''(x_0)$$
$$p_n'''(x_0) = 6a_3 = f'''(x_0)$$
$$|$$
$$p_n^{(k)}(x_0) = k!\,a_k = f^{(k)}(x_0)$$
$$\Longrightarrow a_k = \frac{f^{(k)}(x_0)}{k!}$$

$$p_n(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2}(x - x_0)^2 + \dots$$

Theorem If f has n derivatives at x_0, and $f^{(n)}$ is continuous,

$$\lim_{x \to x_0} \frac{f(x) - p_n(x)}{(x - x_0)^n} = 0$$

Pf. $\lim = \lim \dfrac{f'(x) - p_n'(x)}{n(x - x_0)^{n-1}} = \dots$

$$= \lim \frac{f^{n}(x) - p_n^{(n)}(x)}{n!}$$

Exemples

- $f(x) = e^x$
- $f(x) = \frac{1}{1-x}$
- $f(x) = \frac{1}{1+x}$
- $f(x) = \ln\left(\frac{1}{1+x}\right)$
- $f(x) = \cos(x)$
- $f(x) = \frac{1}{x}$
- $f(x) = \frac{1}{1+x^2}$
- $f(x) = \arctan(x)$

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Big O notation:

We say $f(x) = O(g(x))$ when $x \to x_0$ if

$$|f(x)| \le M\, g(x) \quad \text{for} \quad x_0 - \delta \le x \le x_0 + \delta$$

for some $M, \delta > 0$

<u>Taylor's theorem</u>

If $f: \mathbb{R} \to \mathbb{R}$ is differentiable at x_0 d times with continuous derivatives, then

$$f(x) = \underbrace{f(x_0) + f'(x_0)(x - x_0) + \dots + \frac{f^{(d)}(x_0)}{d!}(x - x_0)^d}_{pd(x)} + O((x - x_0)^{d+1})$$

because

$$f(x) - pd(x) = \frac{f^{(d+1)}(x_0)}{(d+1)!}(x - x_0)^{d+1} + r(x)$$

and $\frac{r(x)}{(x - x_0)^{d+1}} \xrightarrow{x \to x_0} 0$

We can use this to compute limits:

- $$\lim_{x \to 0} \frac{\sin x}{x} = \lim_{x \to 0} \frac{x - \frac{x^3}{6!} + O(x^4)}{x} = \lim_{x \to x_0} \frac{1 - \overset{\to 0}{\frac{x^2}{6!}} + \frac{1}{x} \overset{\le Mx^4}{O(x^4)}}{1}$$

$$\sin x = x - \frac{x^3}{6} + \_\_$$

- $$\lim_{x \to 0} \frac{1 - \cos x}{x} = \lim_{x \to 0} \frac{1 - (1 + O(x^2))}{x} = \lim_{x \to 0} \frac{O(x^2)}{x} = 0$$

$$\cos x = 1 - \frac{x^2}{2}$$

If $f = O(x^2)$, $|f(x)| \le M x^2$

- $\lim\limits_{x\to 0} \frac{1-\cos x}{x^2} = \lim\limits_{x\to 0} \frac{\frac{x^2}{2} + O(x^4)}{x^2} = \lim\limits_{x\to 0}\left(\frac{1}{2} + \underbrace{\frac{O(x^4)}{x^2}}_{\to 0}\right) = \frac{1}{2}$

$\cos x = 1 - \frac{x^2}{2!} + O(x^4)$

- $\lim\limits_{x\to 0} \frac{\sin x - x}{x^3} = \lim\limits_{x\to 0} \frac{-\frac{x^3}{6} + O(x^5)}{x^3} = -\frac{1}{6}$

$\sin x = x - \frac{x^3}{6} + O(x^5)$

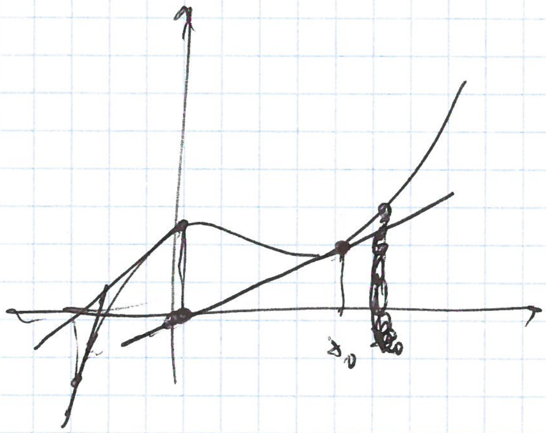
- $\lim\limits_{x\to 0} \frac{x^2 e^x}{\cos x - 1} = \lim\limits_{x\to 0} \frac{x^2 + x^3 + O(x^4)}{-\frac{x^2}{2} + O(x^4)} = \lim\limits_{x\to 0} \frac{1 + x + O(x^2)}{-\frac{1}{2} + O(x^2)} = 2$

$x^2 e^x = x^2\left(1 + x + \frac{x^2}{2} + \ldots\right) = x^2 + x^3 + O(x^4)$

$\cos x - 1 = \frac{x^2}{2} + O(x^4)$

Newton's method

Want to solve $f(x) = 0$.



Start by x_0,

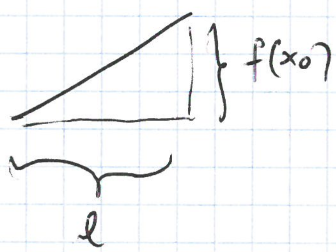
Define

$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$

$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$

$\vdots$

$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$



$f(x_0) = f'(x_0)\cdot \ell$